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EQUATIONS OF GEODESIC WITH AN APPROXIMATE INFINITE SERIES (α, β)-METRIC

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Abstract. In the present paper, we consider the condition that is a geodesic equation on a Finsler space with an (α, β)-metric. Next we find the conditions that are equations of geodesic on the Finsler space with an approximate infinite series (α, β)-metric.

1. Introduction

A Finsler metric $L(\alpha, \beta)$ in a differentiable manifold M^n is called an (α, β)-metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ and a one-form $\beta = b_i(x)y^i$ on M^n.

The geodesics of a two-dimensional Finsler space $F^2 = (M^2, L)$ with an (α, β)-metric are regarded as the curves of the associated Riemannian space $R^2 = (M^2, \alpha)$ which are bent by the differential 1-form β (cf. [10]). M. Matsumoto and H. S. Park [11] have expressed the differential equations of geodesics in two-dimensional Finsler spaces with a Randers metric and a Kropina metric in the most clean form $y'' = f(x, y, y')$, respectively.

Let F^n be an n-dimensional Finsler space with the fundamental function $L(x, y)$ and the fundamental tensor $g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j L^2/2$. The tangent vector space F^n_x with the origin removed at every point x of F^n is a Minkowski space with the norm $L(x, y)$. On the other hand, F^n_x is also regarded as a Riemannian space with the fundamental quadratic form $ds^2 = g_{ij} dy^i dy^j$ [14], as it is often emphasized in [6]. Therefore the concept of geodesic is introduced in the Riemannian space F^n_x by applying

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to F^n_x the usual theory of calculus of variations, and a geodesic coincides with an autoparallel curve with respect to the Riemannian connection.

In the present paper, we consider the conditions that the Finsler space with an (α,β)-metric be geodesic. Next we find the conditions that the Finsler space with an approximate infinite series (α,β)-metric be equations of geodesic.

2. Priliminaries

We consider a Finsler space $F^n = (M, L)$ with an (α,β)-metric. Then α is a Riemannian metric and β is a 1-form in (y^i) as follows:

$$\alpha^2 = a_{ij}(x)y^iy^j \quad \text{and} \quad \beta = b_i(x)y^i.$$

The space $R^n = (M,\alpha)$ is called the *associated Riemannian space* of F^n. The regularity of α is supposed and we denote by (a^{ij}) the inverse of (a_{ij}).

Throughout the present paper, we use the following notation as follows:

For a function $L(\alpha,\beta)$ we put

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}, \quad \text{etc.}$$

For instance, we have $L_{\alpha\alpha} + L_{\beta\beta} = L$ from the homogeneity of L.

The subscripts i, j, $\cdots$, are used to denote $\dot\partial_i$, $\dot\partial_j$.

For instance, $\alpha^2 = a_{rs}(x)y^ry^s$ yields

$$\alpha\alpha_i = a_{ir}y^r, \quad \alpha\alpha_{ij} + \alpha_i\alpha_j = a_{ij}, \quad \beta_i = b_i.$$

If we put $a_{ir}y^r = Y_i$ and $a^{ir}b_r = B^i$, then

$$\alpha\alpha_{ij} = a_{ij} - \frac{Y_iY_j}{\alpha^2} = k_{ij}$$

are components of the angular metric tensor of R^n.

Throughout the following we are concerned with the Levi-Civita connection $\gamma = ({\gamma_j}^i{}_k(x))$ of R^n. On account of [1], we get

$${\gamma_j}^i{}_k = \frac{1}{2}a^{ir}(\partial_k a_{jr} + \partial_j a_{kr} - \partial_r a_{jk}),$$

and denote by $(,)$ the covariant differentiation with respect to γ.

From γ a pair connection ${}^*\gamma = ({\gamma_j}^i{}_k, {\gamma_0}^i{}_j, 0)$ is induced in F^n. The h-covariant differentiation with respect to ${}^*\gamma$ is also denote by $(;)$.

Let us list the symbols in F^n for the later use:

(a) $\quad r_{ij} = (b_{i,j} + b_{j,i})/2, \quad s_{ij} = (b_{i,j} - b_{j,i})/2,$

$$\text{(2.1)}\qquad \text{(b)}\quad r^i{}_j = a^{ir} r_{rj}, \quad s^i{}_j = a^{ir} s_{rj},$$
$$\text{(c)}\quad r_i = b_r r^r{}_i = B^r r_{ri}, \quad s_i = b_r s^r{}_i = B^r s_{ri}.$$

It is noted that $s_{ij} = (\partial_j b_i - \partial_i b_j)/2$ and $s_r B^r = 0$.

Let $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ be the Berwald connection of F^n and put

$$\text{(2.2)}\qquad \begin{aligned} &2G^i = \gamma_0{}^i{}_0 + 2D^i, \quad G^i{}_j = \gamma_0{}^i{}_j + D^i{}_j,\\ &G_j{}^i{}_k = \gamma_j{}^i{}_k + D_j{}^i{}_k, \end{aligned}$$

where $D^i{}_j = \dot\partial_j D^i$ and $D_j{}^i{}_k = \dot\partial_k D^i{}_j$. Berwald connection $B\Gamma$ [6] is uniquely determined by the system of axioms given in [13]:

(1) L-metrical, (2) $G_j{}^i{}_k - G_k{}^i{}_j = 0$,

(3) $\dot\partial_k G^i{}_j - G_k{}^i{}_j = 0$, (4) $y^r G_r{}^i{}_j - G^i{}_j = 0$.

Among these axioms (2)$\sim$(4) have been satisfied by the quantities given in the right-hand sides of (2.2). Thus we have to treat of (1) alone, which is written as

$$L_{;i} = \partial_i L - G^r{}_i \dot\partial_r L = L_{/i} - D^r{}_i L_r = 0.$$

Since we have

$$\begin{aligned} L_{,i} &= L_\alpha \alpha_i + L_\beta \beta_{,i} = L_\beta b_{r,i} y^r,\\ L_r &= L_\alpha \alpha_r + L_\beta \beta_r = L_\alpha Y_r/\alpha + L_\beta b_r, \end{aligned}$$

$L_{;i} = 0$ is written in the form

$$\text{(2.3)}\qquad (L_\alpha Y_r + \alpha L_\beta b_r) D^r{}_i = \alpha L_\beta b_{r,i} y^r.$$

Next, we shall consider the two-dimensional case. Let us denote by $R(C) = 0$ the differential equation of the Weierstrass form of a geodesic C of R^2. $R(C)$ is given by

$$R(C) = \alpha_{\alpha(\beta)} - \alpha_{\beta(\alpha)} + (y^1 \dot y^2 - y^2 \dot y^1) W_r,$$

where $\alpha_i = \partial\alpha/\partial x^i$ and $\alpha_{(i)} = \partial\alpha/\partial y^i$, $y^i = dx^i/dt$ and $\dot y^i = dy^i/dt$ and W_r is the Weierstrass invariant of R^2 (cf. [11]) By putting $y^i{}_{;0} = \dot y^i + \gamma_0{}^i{}_0$, $R(C)$ can be written in the form

$$\text{(2.4)}\qquad R(C) = (y^1 y^2_{;0} - y^2 y^1_{;0}) W_r, \quad W_r = \{a_{11}a_{22} - (a_{12})^2\}/\alpha^3.$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For example, $\gamma_0{}^i{}_0$ is $hp(2)$.

Then we have

LEMMA 2.1 ([11]). *In a two-dimensional Finsler space with (α,β)-metric $L(\alpha,\beta)$, the geodesics are given by the differential equation*

$$(L_\alpha + w\alpha\gamma^2)R(C) + \beta_{;r}y^r\delta\omega - L_\beta(b_{1;2} - b_{2;1}) = 0,$$

where w is the intrinsic Weierstrass invariant, $R(C)$ is defined by (2.4) *and $\delta = (a_{1r}b_2 - a_{2r}b_1)y^r$.*

Suppose that the Riemannian metric α be positive-definite. Then we may refer to an isothermal coordinate system $(x^i, y^i) = (x, y, \dot{x}, \dot{y})$ ([3]) such that

$$\alpha = aE, \quad a = a(x,y) > 0, \quad E = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{1 + y'^2}.$$

Then $R(C)$ is of the form $R_i(C)$, where $R_i(C) = \frac{a}{E^3}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{1}{E}(a_x\dot{y} - a_y\dot{x})$. Next $\gamma^2 = (b_1\dot{y} - b_2\dot{x})^2$, and hence we may put $\gamma = b_1\dot{y} - b_2\dot{x}$ ([3]) and $\delta = -a^2\gamma$. Therefore, we have

LEMMA 2.2 ([11]). *For the Finsler space of Lemma 2.1, if α is positive-definite and we refer to an isothermal coordinate system (x,y) such that $\alpha = aE$, then the differential equation of a geodesic is of the form:*

$$\begin{aligned}\{L_\alpha + aE\omega(b_1\dot{y} - b_2\dot{x})^2\}\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\}\\ -E^3L_\beta(b_{1y} - b_{2x}) - E^3a^2\omega(b_1\dot{y} - b_2\dot{x})b_{0;0} = 0,\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}b_{0;0} = {}&(b_{1x}\dot{x} + b_{1y}\dot{y})\dot{x} + (b_{2x}\dot{x} + b_{2y}\dot{y})\dot{y}\\ &+ \frac{1}{a}\{(\dot{x}^2 + \dot{y}^2)(a_xb_1 + a_yb_2) - 2(b_1\dot{x} + b_2\dot{y})(a_x\dot{x} + a_y\dot{y})\}\end{aligned} \tag{2.6}$$

and we put $b_{ix} = \partial b_i/\partial x$, $b_{iy} = \partial b_i/\partial y$, $a_x = \partial a/\partial x$ and $a_y = \partial a/\partial y$.

Let us consider the *r-th series (α,β)-metric*

$$L(\alpha,\beta) = \beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k, \tag{2.7}$$

where we assume $\alpha < \beta$.

Then the metric above is called an *approximate infinite series (α,β)-metric* or the *rth approximate infinite series (α,β)-metric.*

If $r = 1$, then $L = \alpha + \beta$ is a Randers metric. The condition that the Randers space be a Berwald space, and a Douglas space are found in [12], respectively. If $r = 2$, then $L = \alpha + \beta + \frac{\alpha^2}{\beta}$ is treated in [8] as an

(α,β)-metric that a locally Minkowski space is flat-parallel. If $r=\infty$, then this metric (2.7) is expressed as the form

$$L(\alpha,\beta)=\frac{\beta^2}{\beta-\alpha}. \tag{2.8}$$

Then the metric above is called an *infinite series* (α,β)*-metric*.

3. Equation of geodesic of (α,β)-metric

In the present paper, we find the function $G^i(x,y)$ appearing in the equations of geodesic of a Finsler space with (α,β)-metric, that is, solve $D_j{}^i{}_k$ with (2.3). It is rewritten in the form

$$L_\beta(r_{i0}-s_{i0})=\ell_r D^r{}_i, \tag{3.1}$$

in the notation of (2.1), because we have $\ell_i=L_\alpha Y_i/\alpha+L_\beta b_i$. Then we have

$$L_\beta r_{00}=2\ell_r D^r. \tag{3.2}$$

If we differentiate this by y^i and paying attention to $L_{\beta\alpha}\alpha_i+L_{\beta\beta}b_i=L_{\beta\beta}p_i$, where $p_i=b_i-(\beta/\alpha^2)Y_i$, then we have

$$L_{\beta\beta}p_i r_{00}+2L_\beta r_{i0}=\frac{2h_{ri}D^r}{L}+2\ell_r D^r{}_i.$$

Since we have [2], that is,

$$h_{ij}=\left(\frac{LL_\alpha}{\alpha}\right)\left(a_{ij}-\frac{Y_iY_j}{\alpha^2}\right)+LL_{\beta\beta}p_ip_j,$$

the substitution in the above yields

$$D^i=\left(\frac{\eta}{\alpha^2}\right)y^i+\left(\frac{\alpha L_{\beta\beta}}{2L_\alpha}\right)(r_{00}-2\xi)p^i+\left(\frac{\alpha L_\beta}{L_\alpha}\right)s^i{}_0, \tag{3.3}$$

where $\eta=Y_rD^r$ and $\xi=p_rD^r$ and $p^i=a^{ir}p_r=b^i-(\beta/\alpha^2)y^i$.

We shall find η and ξ. First (3.2) may be written as

$$L_\beta r_{00}=2\left(\frac{L_\alpha Y_r}{\alpha}+L_\beta b_r\right)D^r=\left(\frac{2L_\alpha}{\alpha}\right)\eta+2L_\beta b_rD^r. \tag{3.4}$$

Next we have

$$\xi=\left\{b_r-\left(\frac{\beta}{\alpha^2}\right)Y_r\right\}D^r=b_rD^r-\left(\frac{\beta}{\alpha^2}\right)\eta.$$

Eliminating $b_r D^r$ from these equations, we get

$$\eta = \left(\frac{\alpha^2 L_\beta}{2L}\right)(r_{00} - 2\xi). \tag{3.5}$$

Further, multiplying by p_i, (3.3) yields

$$\xi = \left(\frac{\gamma^2 L_{\beta\beta}}{2\alpha L_\alpha}\right)(r_{00} - 2\xi) + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s_0.$$

On account of $L_{\beta\beta} = (\alpha/\beta)^2 L_{\alpha\alpha}$, we have T of [5] in the form

$$T = \left(\frac{L}{\alpha}\right)^3 \left(L_\alpha + \frac{\gamma^2 L_{\beta\beta}}{\alpha}\right), \quad \gamma^2 = b^2\alpha^2 - \beta^2. \tag{3.6}$$

Hence, the above yields

$$\xi = \left(\frac{L^3}{\alpha^2 T}\right)\left\{\left(\frac{\gamma^2 L_{\alpha\alpha}}{2\beta^2}\right) r_{00} + L_\beta s_0\right\}. \tag{3.7}$$

Consequently, (3.5) and (3.7) give η and ξ, and hence (3.3) can be rewritten in the form

$$D^i = \left(\frac{\eta}{\alpha^2}\right)\left\{y^i + \left(\frac{\alpha^3 L L_{\alpha\alpha}}{\beta^2 L_\alpha L_\beta}\right) p^i\right\} + \left(\frac{\alpha L_\beta}{L_\alpha}\right) {s^i}_0. \tag{3.8}$$

Therefore, for a Finsler space with (α, β)-metric, the functions $G^i(x, y)$ are of the form $2G^i = {\gamma_0}^i{}_0 + 2D^i$, where ${\gamma_j}^i{}_k$ are Christoffel symbols of the associated Riemannian space and D^i are given by (3.8) with (3.5) and (3.7).

Thus G^i are obtained without use of the inverse fundamental tensor g^{ij}, similarly to the case of dimension two [11].

We have, of course, the general equations of geodesic C of F^n in the form

$$\frac{d^2x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where s is the arc-length of C in F^n.

For a Finsler space $F^n = (M, L(\alpha, \beta))$, it may be convenient to write the equations with the Riemannian paramenter $\sigma : d\sigma^2 = \alpha^2(x, dx)$. Owing to [4], we have the equations in the form

$$\frac{d^2x^i}{d\sigma^2} + 2G^i\left(x, \frac{dx}{d\sigma}\right) = -\left\{\frac{\sigma''}{(\sigma')^2}\right\}\frac{dx^i}{d\sigma}, \tag{3.9}$$

where $\sigma' = d\sigma/ds$. We observe

$$\sigma' = \frac{1}{L(x, x')},$$

where we put $x' = dx/d\sigma$. It follows that

$$\sigma'' = \frac{d\sigma'}{ds} = -\left\{\frac{1}{L^3(x, x')}\right\}\left\{L_\alpha\left(\frac{d\alpha}{d\sigma}\right) + L_\beta\left(\frac{d\beta}{d\sigma}\right)\right\}.$$

Since $\alpha(x, dx/d\sigma) = 1$ along C, we have $d\alpha/d\sigma = 0$, and

$$\frac{d\beta}{d\sigma} = r_{00} + b_r\gamma_0{}^r{}_0 + G,$$

where $G = b_i\left(\frac{d^2x^i}{d\sigma^2}\right)$. Consequently,

$$\frac{\sigma''}{(\sigma')^2} = -\left(\frac{L_\beta}{L}\right)(r_{00} + b_r\gamma_0{}^r{}_0 + G),$$

where $y^i = dx^i/d\sigma$. Thus (3.9) may be written

$$\frac{d^2x^i}{d\sigma^2} + \gamma_0{}^i{}_0 + 2D^i = \left(\frac{L_\beta}{L}\right)(r_{00} + b_r\gamma_0{}^r{}_0 + G)\left(\frac{dx^i}{d\sigma}\right). \tag{3.10}$$

To eliminate G, we multiply by b_i and (3.10) gives

$$G + b_i\gamma_0{}^i{}_0 + 2b_iD^i = \left(\frac{L_\beta\beta}{L}\right)(r_{00} + b_r\gamma_0{}^r{}_0 + G).$$

Substituting from (3.10), the left-hand side can be written as

$$G + b_i\gamma_0{}^i{}_0 + r_{00} = \left(\frac{2L_\alpha}{L_\beta}\right)\eta.$$

Hence, $G + b_i\gamma_0{}^i{}_0 + r_{00} = 2\eta L_\alpha/L_\beta$. Therefore (3.10) is written in the form

$$\begin{aligned} &\frac{d^2x^i}{d\sigma^2} + \gamma_j{}^i{}_k(x)\left(\frac{dx^j}{d\sigma}\right)\left(\frac{dx^k}{d\sigma}\right) \\ &+ \left(\frac{2\eta LL_{\alpha\alpha}}{\beta^2L_\alpha L_\beta}\right)p^i + \left(\frac{2L_\beta}{L_\alpha}\right)s^i{}_j\left(\frac{dx^j}{d\sigma}\right) = 0. \end{aligned} \tag{3.11}$$

Therefore we have

THEOREM 3.1. *In a Finsler space $F^n = (M, L(\alpha, \beta))$ with (α, β)-metric, the differential equations of a geodesic C are written in terms of the arc-length σ of C in the associated Riemannian space $R^n = (M, \alpha)$, as (3.11), where $y^i = dx^i/d\sigma$.*

4. Geodesic equation of dimension n with an approximate infinite series (α, β)-metric

In the present section, we consider the conditions that a Finsler space F^n with an approximate infinite series (α, β)-metric be the differential equations of a geodesic. The metric of F^n is (2.7). In this case we have

$$(4.1)\qquad \begin{aligned} L_\alpha &= \sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1}, & L_\beta &= -\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}, \\ L_{\alpha\alpha} &= \frac{1}{\beta}\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}, & L_{\beta\beta} &= \frac{1}{\beta}\sum_{k=0}^{r}(k-1)k\left(\frac{\alpha}{\beta}\right)^{k}. \end{aligned}$$

Now we shall divide our consideration in two cases of which r is even or odd.

(1) Case of $r = 2h$, where h is a positive integer.
When $r = 2h$, we have

$$(4.2)\qquad \begin{aligned} \sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^{k} &= \frac{1}{\beta^{2h}}\sum_{k=0}^{2h}\alpha^k\beta^{2h-k}, \\ \sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1} &= \frac{\beta}{\beta^{2h}}\sum_{k=0}^{2h}k\alpha^{k-1}\beta^{2h-k}, \\ \sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k} &= \frac{1}{\beta^{2h}}\sum_{k=0}^{2h}(k-1)\alpha^k\beta^{2h-k}, \\ \sum_{k=0}^{r}k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2} &= \frac{1}{\beta^{2h-2}}\sum_{k=0}^{2h}(k-1)k\alpha^{k-2}\beta^{2h-k}, \\ \sum_{k=0}^{r}(k-1)k\left(\frac{\alpha}{\beta}\right)^{k} &= \frac{1}{\beta^{2h}}\sum_{k=0}^{2h}(k-1)k\alpha^k\beta^{2h-k}. \end{aligned}$$

Separating the rational and irrational parts in y^i with respect to (4.2), we obtain

$$\begin{aligned} \sum_{k=0}^{2h}\alpha^k\beta^{2h-k} &= I + \alpha J, \\ \sum_{k=0}^{2h}k\alpha^{k-1}\beta^{2h-k} &= M + \alpha K, \end{aligned}$$

$$\sum_{k=0}^{2h}(k-1)\alpha^k\beta^{2h-k} = L+\alpha^3 N, \tag{4.3}$$
$$\sum_{k=0}^{2h}(k-1)k\alpha^{k-2}\beta^{2h-k} = P+\alpha Q,$$
$$\sum_{k=0}^{2h}(k-1)k\alpha^{k}\beta^{2h-k} = R+\alpha S,$$

where

$$I=\sum_{k=0}^{h}\alpha^{2k}\beta^{2h-2k}, \qquad J=\sum_{k=0}^{h-1}\alpha^{2k}\beta^{2h-2k-1},$$
$$K=\sum_{k=1}^{h}2k\alpha^{2k-2}\beta^{2h-2k}, \qquad L=\sum_{k=0}^{h}(2k-1)\alpha^{2k}\beta^{2h-2k},$$
$$M=\sum_{k=1}^{h-1}(2k+1)\alpha^{2k}\beta^{2h-2k-1}, \qquad N=\sum_{k=1}^{h-1}2k\alpha^{2k-2}\beta^{2h-2k-1},$$
$$P=\sum_{k=0}^{h}(2k-1)2k\alpha^{2k-2}\beta^{2h-2k}, \qquad Q=\sum_{k=0}^{h-1}2k(2k+1)\alpha^{2k-2}\beta^{2h-2k-1},$$
$$R=\sum_{k=0}^{h}(2k-1)\alpha^{2k}\beta^{2h-2k}, \qquad S=\sum_{k=0}^{h-1}2k(2k+1)\alpha^{2k}\beta^{2h-2k-1}.$$

Substituting (4.1), (4.2) and (4.3) into (3.6), we have

$$T=\frac{(I+\alpha J)^3\Omega}{\alpha^4\beta^{8h-2}}, \tag{4.4}$$

where $\Omega=\alpha\beta^2(M+\alpha K)+\gamma^2(R+\alpha S)$.

Substituting (4.1), (4.2) and (4.3) into (3.7), we get

$$\xi=\alpha^2\big\{\gamma^2(P+\alpha Q)r_{00}-2\beta(L+\alpha^3 N)s_0\big\}/2\Omega. \tag{4.5}$$

Further substituting (4.1), (4.2) and (4.3) into (3.5), we obtain

$$\begin{aligned}\eta=-\frac{\alpha^2(L+\alpha^3N)}{2\beta(I+\alpha J)\Omega}\big[&\{\alpha\beta^2(M+\alpha K)+\gamma^2(R+\alpha S)\\ &-\alpha^2\gamma^2(P+\alpha Q)\}r_{00}+2\alpha^2\beta(L+\alpha^3N)s_0\big].\end{aligned} \tag{4.6}$$

Furthermore substituting (4.1), (4.2), (4.3) and (4.6) into (3.8), we have

$$
\begin{aligned}
D^i = & -\frac{(L+\alpha^3 N)}{2\beta(I+\alpha J)\Omega}\big[\{\alpha\beta^2(M+\alpha K)+\gamma^2(R+\alpha S) \\
& -\alpha^2\gamma^2(P+\alpha Q)\}r_{00}+2\alpha^2\beta(L+\alpha^3 N)s_0\big] \\
& \left\{y^i-\frac{\alpha^3(I+\alpha J)(P+\alpha Q)}{\beta(M+\alpha K)(L+\alpha^3 N)}P^i\right\}-\frac{\alpha(L+\alpha^3 N)}{\beta(M+\alpha K)}s^i{}_0.
\end{aligned} \tag{4.7}
$$

Therefore we have

Theorem 4.1. *For a Finsler space with an approximate infinite series (α,β)-metric, the functions $G^i(x,y)$ are of the form $2G^i=\gamma_0{}^i{}_0+2D^i$, where $\gamma_j{}^i{}_k$ are Christoffel symbols of the associated Riemannian space and D^i are given by (4.7) with (4.5) and (4.6).*

Next, paying attenting to $G+b_i\gamma_0{}^i{}_0+r_{00}=\alpha^2(M+\alpha K)[\{\alpha\beta^2(M+\alpha K)+\gamma^2(R+\alpha S)-\alpha^2\gamma^2(P+\alpha Q)\}r_{00}+2\alpha^2\beta(L+\alpha^3 N)s_0]/(I+\alpha J)\Omega$ and substituting (4.1), (4.2) and (4.3) into (3.11), we get

$$
\begin{aligned}
& \frac{d^2x^i}{d\sigma^2}+\gamma_j{}^i{}_k(x)\left(\frac{dx^j}{d\sigma}\right)\left(\frac{dx^k}{d\sigma}\right)+\frac{\alpha^2(P+\alpha Q)}{\beta^2(M+\alpha K)\Omega} \\
& \Big[\{\alpha\beta^2(M+\alpha K)+\gamma^2(R+\alpha S)-\alpha^2\gamma^2(P+\alpha Q)\}r_{00} \\
& +2\alpha^2\beta(L+\alpha^3 N)s_0\Big]p^i-\frac{2(L+\alpha^3 N)}{\beta(M+\alpha K)}s^i{}_j\left(\frac{dx^j}{d\sigma}\right)=0.
\end{aligned} \tag{4.8}
$$

Therefore we have

Theorem 4.2. *In a Finsler space with an approximate infinite series (α,β)-metric, the differential equations of a geodesic C are written in terms of the arc-length σ of C in the associated Riemannian space $R^n=(M,\alpha)$ as (4.8), where $y^i=dx^i/d\sigma$.*

(2) Case of $r=2h+1$, where h is a positive integer.
When $r=2h+1$, we have

$$
\begin{aligned}
& L=\frac{1}{\beta^{2h}}(\beta I+\alpha U), && L_\alpha=\frac{1}{\beta^{2h}}(O+\alpha\beta K), \\
& L_\beta=-\frac{1}{\beta^{2h+1}}(\beta L+\alpha^3 K), && L_{\alpha\alpha}=\frac{1}{\beta^{2h}}(\beta P+\alpha W), \\
& L_{\beta\beta}=\frac{1}{\beta^{2h+2}}(\beta R+\alpha T),
\end{aligned} \tag{4.9}
$$

where

$$(4.10)\qquad \begin{aligned} O &= \sum_{k=0}^{h}(2k+1)\alpha^{2k}\beta^{2h-2k}, & T &= \sum_{k=0}^{h} 2k(2k+1)\alpha^{2k}\beta^{2h-2k},\\ U &= \sum_{k=0}^{h}\alpha^{2k}\beta^{2h-2k}, & W &= \sum_{k=0}^{h}(2k+1)2k\alpha^{2k-2}\beta^{2h-2k}. \end{aligned}$$

Substituting (4.9) and (4.10) into (3.6), we have

$$(4.11)\qquad T = (\beta I + \alpha U)^3\Omega_1/\alpha^4\beta^{8h+2},$$

where $\Omega_1 = \alpha\beta^2(O + \alpha\beta K) + \gamma^2(\beta R + \alpha T)$.

Substituting (4.9) and (4.10) into (3.7), we get

$$(4.12)\qquad \xi = \alpha^2\{\gamma^2(\beta P + \alpha W)r_{00} - 2\beta(\beta L + \alpha^3 K)s_0\}/2\Omega_1.$$

Further substituting (4.9) and (4.10) into (3.5), we obtain

$$(4.13)\qquad \begin{aligned} \eta = &-\alpha^2(\beta L + \alpha^3 K)\big[\{\Omega_1 - \alpha^2\gamma^2(\beta P + \alpha W)\}r_{00}\\ &+ 2\alpha^2\beta(\beta L + \alpha^3 K)s_0\big]/2\beta(\beta I + \alpha U)\Omega_1. \end{aligned}$$

Furthermore substituting (4.9) and (4.10) into (3.8), we have

(4.14)

$$\begin{aligned} D^i = &\big[\{\Omega_1 - \alpha^2\gamma^2(\beta P + \alpha W)\}r_{00} + 2\alpha^2\beta(\beta L + \alpha^3 K)s_0\big]\\ &\big[\{\beta(O + \alpha\beta K)(\beta L + \alpha^3 K)y^i - \alpha^3(\beta I + \alpha U)(\beta P + \alpha W)p^i\big]\\ &\big/2\beta^2(\beta I + \alpha U)(O + \alpha\beta K)\Omega_1 - \alpha(\beta L + \alpha^3 K){s^i}_0\big/\beta(O + \alpha\beta K). \end{aligned}$$

Therefore we have

Theorem 4.3. *For a Finsler space with an approximate infinite series (α,β)-metric, the functions $G^i(x,y)$ are of the form $2G^i = {\gamma_0}^i{}_0 + 2D^i$, where ${\gamma_j}^i{}_k$ are Christoffel symbols of the associated Riemannian space and D^i are given by (4.14) with (4.12) and (4.13).*

Next, paying attenting to $G + b_i{\gamma_0}^i{}_0 + r_{00} = \alpha^2(O + \alpha\beta K)[\{\Omega_1 - \alpha^2\gamma^2(\beta P + \alpha W)\}r_{00} + 2\alpha^2\beta(\beta L + \alpha^3 K)s_0\big]/(\beta I + \alpha W)\Omega_1$ and substituting

(4.9) and (4.10) into (3.11), we get

(4.15)
$$\begin{aligned}
&\frac{d^2x^i}{d\sigma^2}+\gamma_j{}^i{}_k(x)\left(\frac{dx^j}{d\sigma}\right)\left(\frac{dx^k}{d\sigma}\right)\\
&+\frac{\alpha^2(\beta P+\alpha W)\Big[\{\Omega_1-\alpha^2\gamma^2(\beta P+\alpha W)\}r_{00}+2\alpha^2\beta(\beta L+\alpha^3K)s_0\Big]}{\beta^2(O+\alpha\beta K)\Omega_1}p^i\\
&-\frac{2(\beta L+\alpha^3K)}{\beta(O+\alpha\beta K)}s^i{}_j\left(\frac{dx^j}{d\sigma}\right)=0.
\end{aligned}$$

Therefore we have

Theorem 4.4. *In a Finsler space with an approximate infinite series (α,β)-metric, the differential equations of a geodesic C are written in terms of the arc-length σ of C in the associated Riemannian space $R^n=(M,\alpha)$ as (4.15), where $y^i=dx^i/d\sigma$.*

5. Geodesic equation of dimension two

In the present section, by referring an isothermal coordinate system, we find the differential equations of geodesics of a two-dimensional Finsler space satisfying an approximate infinite series (α,β)-metric (2.7).

Now we shall divide our consideration in two cases of which r is even or odd.

(1) Case of $r=2h$, where h is a positive integer.
When $r=2h$, we have

$$\begin{aligned}
&L=\frac{I+\alpha J}{\beta^{2h-1}},\qquad L_\alpha=\frac{M+\alpha K}{\beta^{2h-1}},\quad L_\beta=-\frac{L+\alpha^3N}{\beta^{2h}},\\
&L_{\alpha\alpha}=\frac{P+\alpha Q}{\beta^{2h-1}},\quad L_{\beta\beta}=\frac{R+\alpha S}{\beta^{2h+1}}.
\end{aligned}\tag{5.1}$$

Substituting (5.1) and $w=(P+\alpha Q)\big/\beta^{2h+1}$ into (2.5), we obtain the differential equations of geodesics as follows:

$$\begin{aligned}
&\{\beta^2(M+\alpha K)+aE(P+\alpha Q)(b_1\dot y-b_2\dot x)^2\}\{a(\dot x\ddot y-\dot y\ddot x)\\
&+E^2(a_x\dot y-a_y\dot x)\}+E\{\beta(L+\alpha^3N)(b_{1y}-b_{2x})\\
&-a^2(P+\alpha Q)(b_1\dot y-b_2\dot x)b_{0;0}\}=0,
\end{aligned}\tag{5.2}$$

where $b_{0;0}$ is given by (2.6).

If we take x of (x,y) as the parameter of curve C, that is, $\dot{x}=1$, $\dot{y}=y'$, $\ddot{x}=0$, $\ddot{y}=y''$ and we put $E^2=1+(y')^2$, then (5.2) is reduced to

$$\begin{aligned}
&\big[\{(b_1+b_2y')^2+a^2E^2Q_1(b_1y'-b_2)^2\}\{a(y'')+E^2(a_xy'-a_y)\}\\
&+\{a^3E^6(b_1+b_2y')N_1(b_{1y}-b_{2x})-a^3E^4Q_1(b_1y'-b_2)b^*_{0;0}\}\big]\\
&+E\big[a(b_1+b_2y')^2K_1+P_1(b_1y'-b_2)^2\}\{a(y'')+E^2(a_xy'-a_y)\}\\
&+\{E^2(b_1+b_2y')L_1(b_{1y}-b_{2x})-a^2E^2P_1(b_1y'-b_2)b^*_{0;0}\}\big]\\
&=0,
\end{aligned}\tag{5.3}$$

where

$$\begin{aligned}
b^*_{0;0}&=(b_{1x}+b_{1y}y')+(b_{2x}+b_{2y}y')y'\\
&\quad+\frac{1}{a}\big\{(1+(y')^2)(a_xb_1+a_yb_2)-2(b_1+b_2y')(a_x+a_yy')\big\},\\
K_1&=\sum_{k=1}^{h}2ka^{2k-2}E^{2k-2}(b_1+b_2y')^{2h-2k},\\
L_1&=\sum_{k=0}^{h}(2k-1)a^{2k}E^{2k}(b_1+b_2y')^{2h-2k},\\
M_1&=\sum_{k=0}^{h-1}(2k+1)a^{2k}E^{2k}(b_1+b_2y')^{2h-2k-1},\\
N_1&=\sum_{k=1}^{h-1}2ka^{2k-2}E^{2k-2}(b_1+b_2y')^{2h-2k-1},\\
P_1&=\sum_{k=0}^{h}(2k-1)2ka^{2k-2}E^{2k-2}(b_1+b_2y')^{2h-2k},\\
Q_1&=\sum_{k=1}^{h-1}2k(2k-1)a^{2k-2}E^{2k-2}(b_1+b_2y')^{2h-2k-1}.
\end{aligned}$$

Since E is irrational in (y'), (5.3) is divided into two equations as follows:

$$\begin{aligned}
&\{(b_1+b_2y')^2M_1+a^2E^2Q_1(b_1y'-b_2)^2\}\{a(y'')+E^2(a_xy'-a_y)\}\\
&+\{a^3E^6(b_1+b_2y')N_1(b_{1y}-b_{2x})-a^3E^4Q_1(b_1y'-b_2)b^*_{0;0}\}=0,
\end{aligned}\tag{5.4}$$

$$\begin{aligned}
&a\{(b_1+b_2y')^2K_1+P_1(b_1y'-b_2)^2\}\{a(y'')+E^2(a_xy'-a_y)\}\\
&+\{E^2(b_1+b_2y')L_1(b_{1y}-b_{2x})-a^3E^2P_1(b_1y'-b_2)b^*_{0;0}\}=0.
\end{aligned}\tag{5.5}$$

Furthermore, (5.4) and (5.5) are rewritten in the form

$$
\begin{aligned}
&a(y'') + \{1 + (y')^2\}(a_x y' - a_y) \\
(5.6)\quad =&\ - a^3\{1 + (y')^2\}^2\big[\{1 + (y')^2\}(b_1 b_2 y')N_1(b_{1y} - b_{2x}) \\
&- Q_1(b_1 y' - b_2)b^*_{0;0}\big]\Big/\big[(b_1 + b_2 y')^2 M_1 \\
&+ a^2\{1 + (y')^2\}Q_1(b_1 y' - b_2)^2\big],
\end{aligned}
$$

$$
\begin{aligned}
&a(y'') + \{1 + (y')^2\}(a_x y' - a_y) \\
(5.7)\quad =&\ - \{1 + (y')^2\}\{(b_1 + b_2 y')L_1(b_{1y} - b_{2x}) - a^2 P_1(b_1 y' - b_2)b^*_{0;0}\} \\
&\Big/a\{(b_1 + b_2 y')^2 K_1 + P_1(b_1 y' - b_2)^2\}.
\end{aligned}
$$

Thus we have

THEOREM 5.1. *Let F^2 be a two-dimensional Finsler space with an approximate infinite series (α, β)-metric* (2.7), *where α is assumed to be positive definite. If we refer to an isothermal coordinate system (x, y) such that $\alpha = aE$ and $E = \sqrt{1 + (y')^2}$ then the differential equations of a geodesic $y = y(x)$ of F^2 are given by* (5.6) *and* (5.7).

Next, we deal with the case where the associated Riemannian space is Euclidean one with an orthonormal coordinate system. Then $a = 1$, $a_x = 0$ and $a_y = 0$. If we take b_1 and b_2 such that $b_1 = \partial b/\partial x$ and $b_2 = \partial b/\partial y$ for a scalar b, then $b_{1y} - b_{2x} = 0$. Thus (5.6) and (5.7) are reduced to

$$(5.6')\quad y'' = \frac{\{1 + (y')^2\}^2 Q_2(b_1 y' - b_2)\{(b_{1x} + b_{1y} y') + (b_{2x} + b_{2y} y')y'\}}{(b_1 + b_2 y')^2 M_2 + \{1 + (y')^2\}Q_2(b_1 y' - b_2)^2},$$

$$(5.7')\quad y'' = \frac{\{1 + (y')^2\}P_2(b_1 y' - b_2)\{(b_{1x} + b_{1y} y') + (b_{2x} + b_{2y} y')y'\}}{(b_1 + b_2 y')^2 K_2 + P_2(b_1 y' - b_2)^2},$$

where

$$
\begin{aligned}
K_2 &= \sum_{k=1}^{h} 2k\{1 + (y')^2\}^{k-1}(b_1 + b_2 y'), \\
M_2 &= \sum_{k=0}^{h-1} (2k + 1)\{1 + (y')^2\}^k (b_1 + b_2 y')^{2h-2k-1}, \\
P_2 &= \sum_{k=0}^{h} (2k - 1)2k\{1 + (y')^2\}^{k-1}(b_1 + b_2 y')^{2h-2k},
\end{aligned}
$$

$$Q_2=\sum_{k=0}^{h-1}(2k-1)2k\{1+(y')^2\}^{k-1}(b_1+b_2y')^{2h-2k-1},$$

$$W_2=\sum_{k=0}^{h}2k(2k+1)\{1+(y')^2\}^{k-1}(b_1+b_2y')^{2h-2k}.$$

Thus we have the following

Corollary 5.2. *Let F^2 be a two-dimensional Finsler space with an approximate infinite series (α,β)-metric (2.7), whose associated Riemannian space $R^2=(M^2,\alpha)$ is Euclidean such that $a=1$ and $a_x=a_y=0$. If we refer to an orthonormal coordinate system (x,y) with respect to α and $b_{1y}-b_{2x}=0$, where $b_1=\partial b/\partial x$, $b_2=\partial b/\partial y$ for a scalar b, then the differential equations of geodesics $y=y(x)$ of F^2 are given by (5.6′) and (5.7′).*

(2) Case of $r=2h+1$, where h is a positive integer.

Substituting (4.9) and $w=(\beta P+\alpha W)/\beta^{2h+2}$ into (2.5), we obtain the differential equations of geodesics as follows:

$$\tag{5.8}\begin{aligned}&\{\beta^2(O+\alpha\beta K)+aE(\beta P+\alpha W)(b_1\dot y-b_2\dot x)^2\}\{a(\dot x\ddot y-\dot y\ddot x)\\&+E^2(a_x\dot y-a_y\dot x)\}+E^3\beta(\beta L+\alpha^3K)(b_{1y}-b_{2x})\\&-E^3a^2(\beta P+\alpha W)(b_1\dot y-b_2\dot x)b_{0;0}=0,\end{aligned}$$

where $b_{0;0}$ is given by (2.6).

If we take x of (x,y) as the parameter of curve C, that is, $\dot x=1$, $\dot y=y'$, $\ddot x=0$, $\ddot y=y''$ and we put $E^2=1+(y')^2$, then (5.8) is reduced to

$$\tag{5.9}\begin{aligned}&\Big[\{(b_1+b_2y')^2O_1+a^2E^2(b_1y'-b_2)^2W_1\}\{ay''+E^2(a_xy'-a_y)\}\\&+E^4\{a^3E^2(b_1+b_2y')(b_{1y}-b_{2x})K_1-a^3(b_1y'-b_2)W_1b^*_{0;0}\}\Big]\\&+E\Big[\{a(b_1+b_2y')^3K_1+a(b_1+b_2y')(b_1y'-b_2)^2P_1\}\{ay''\\&+E^2(a_xy'-a_y)\}+E^2\{(b_1+b_2y')^2(b_{1y}-b_{2x})L_1\\&-a^2\{(b_1+b_2y')(b_1y'-b_2)P_1b^*_{0;0}\}\Big]=0,\end{aligned}$$

where

$$O_1=\sum_{k=0}^{h}(2k+1)a^{2k}E^{2k}(b_1+b_2y')^{2h-2k},$$

$$W_1 = \sum_{k=0}^{h}(2k)(2k+1)a^{2k-2}E^{2k-2}(b_1+b_2y')^{2h-2k}.$$

Since E is irrational in (y'), (5.9) is divided into equations as follows:

$$\begin{aligned}&\{(b_1+b_2y')^2O_1+a^2E^2(b_1y'-b_2)^2W_1\}\{ay''+E^2(a_xy'-a_y)\}\\&+E^4\{a^3E^2(b_1+b_2y')(b_{1y}-b_{2x})K_1-a^3(b_1y'-b_2)W_1b^*_{0;0}\}\\&=0,\end{aligned}\tag{5.10}$$

$$\begin{aligned}&\{a(b_1+b_2y')^3K_1+a(b_1+b_2y')(b_1y'-b_2)^2P_1\}\{ay''\\&+E^2(a_xy'-a_y)\}+E^2\{(b_1+b_2y')^2(b_{1y}-b_{2x})L_1\\&-a^2(b_1+b_2y')(b_1y'-b_2)P_1b^*_{0;0}\}=0.\end{aligned}\tag{5.11}$$

Furthermore, (5.10) and (5.11) are rewritten in the form

$$\begin{aligned}&ay''+\{1+(y')^2\}(a_xy'-a_y)\\=&-\Big\{\{1+(y')^2\}^2\big[a^3\{1+(y')^2\}(b_1+b_2y')(b_{1y}-b_{2x})K_1\\&-a^3(b_1y'-b_2)W_1b^*_{0;0}\big]\Big\}\Big/\Big\{(b_1+b_2y')^2O_1+a^2\{1+(y')^2\}\\&(b_1y'-b_2)^2W_1\Big\},\end{aligned}\tag{5.12}$$

$$\begin{aligned}&ay''+\{1+(y')^2\}(a_xy'-a_y)\\=&-\Big\{\{1+(y')^2\}\big[(b_1+b_2y')^2(b_{1y}-b_{2x})L_1\\&-a^2\{(b_1+b_2y')(b_1y'-b_2)P_1b^*_{0;0}\}\big]\Big\}\Big/\Big\{a(b_1+b_2y')^3K_1\\&+a(b_1+b_2y')(b_1y'-b_2)^2P_1\Big\}.\end{aligned}\tag{5.13}$$

Thus we have

THEOREM 5.3. *Let F^2 be a two-dimensional Finsler space with an approximate infinite series (α,β)-metric (2.7), where α is assumed to be positive definite. If we refer to an isothermal coordinate system (x,y) such that $\alpha = aE$ and $E=\sqrt{1+(y')^2}$, then the differential equations of a geodesic $y=y(x)$ of F^2 are given by (5.12) and (5.13).*

Next, we deal with the case where the associated Riemannian space is Euclidean one with an orthonormal coordinate system. Then $a=1$, $a_x=0$ and $a_y=0$. If we take b_1 and b_2 such that $b_1=\partial b/\partial x$ and

$b_2 = \partial b/\partial y$ for a scalar b, then $b_{1y} - b_{2x} = 0$. Thus (5.12) and (5.13) are reduced to

(5.12′)
$$y'' = \frac{\{1+(y')^2\}^2(b_1y'-b_2)W_2\{(b_{1x}+b_{1y}y')+(b_{2x}+b_{2y}y')y'\}}{(b_1+b_2y')^2O_2+\{1+(y')^2\}(b_1y'-b_2)W_2},$$

(5.13′)
$$y'' = \Big\{\{1+(y')^2\}(b_1+b_2y')(b_1y'-b_2)P_2\{(b_{1x}+b_{1y}y') + (b_{2x}+b_{2y}y')y'\}\Big\}\Big/\Big\{(b_1+b_2y')^3K_2+(b_1+b_2y')(b_1y'-b_2)^2P_2\Big\},$$

where

$$K_2 = \sum_{k=1}^{h} 2k\{1+(y')^2\}^{k-1}(b_1+b_2y')^{2h-2k},$$

$$P_2 = \sum_{k=0}^{h}(2k-1)2k\{1+(y')^2\}^{k-1}(b_1+b_2y')^{2h-2k},$$

$$O_2 = \sum_{k=0}^{h}(2k+1)\{1+(y')^2\}^{k}(b_1+b_2y')^{2h-2k},$$

$$W_2 = \sum_{k=0}^{h} 2k(2k+1)\{1+(y')^2\}^{k-1}(b_1+b_2y')^{2h-2k}.$$

Thus we have the following

Corollary 5.4. *Let F^2 be a two-dimensional Finsler space with an approximate infinite series (α,β)-metric (2.7) whose associated Riemannian space $R^2 = (M^2,\alpha)$ is Euclidean such that $a = 1$ and $a_x = a_y = 0$. If we refer to an orthonormal coordinate system (x,y) with respect to α and $b_{1y} - b_{2x} = 0$, where $b_1 = \partial b/\partial x$, $b_2 = \partial b/\partial y$ for a scalar b, then the differential equations of geodesics $y = y(x)$ of F^2 are given by (5.12′) and (5.13′).*

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